

## Separable Expansion of Potential in Multidimensional Space. Three-Body Problem

### 1. INTRODUCTION

In the present paper the problem of the three-particle bound state is solved by the method of separable expansion of the total potential  $V = V_{12} + V_{23} + V_{31}$  in the six-dimensional space. In a concrete calculation for a sum of two-body potentials of the Gaussian type, a rapid convergence of the proposed procedure is established. The wavefunction of the system considered is presented as a series in matrix elements of the potential  $V$ , which are explicitly calculated in the given case.

In the last few years considerable progress has been achieved in the description of a system consisting of three particles interacting via a short-range potential. In this case, as is well known, equations with Fredholm kernels have been derived [1-4] and rather rapidly convergent regular procedures have been developed for solving three-particle equations [5-13].

However, calculations of the properties of such a system with realistic interaction potentials as the Reid potential are still rather complicated and for positive energies of three-body systems, these only now being developed [14, 15]. This situation is due to the following. First, realistic potentials, being very complicated, generate many components in the wavefunction of a three-particle system. Second, the solution of the Faddeev equations even with central local potentials is still more complicated than that of a two particle system with the same interaction.

Thus, if the first difficulty cannot, generally, be avoided one may try to simplify the solution of three-particle equations with central potentials.

In this note we propose a version of such an attempt based on the separable expansion of the potential  $V = V_{12} + V_{23} + V_{31}$  in the six-dimensional space. The general theory of the method proposed is given in the following section. In Section 3 we give numerical calculations for the case of a sum of two-body potentials of the Gaussian type. Finally some conclusions are given in Section 4.

### 2. SEPARABLE EXPANSION

Let us take the Lippmann-Schwinger equation describing a bound state of three identical spinless particles:

$$|\Psi\rangle = G_0(E) V |\Psi\rangle. \tag{1}$$

In the momentum representation it has the form

$$\psi(\mathbf{qp}) \equiv \langle \mathbf{qp} | \psi \rangle = G_0(qpE) \int d\mathbf{q}' d\mathbf{p}' \langle \mathbf{qp} | V | \mathbf{q}'\mathbf{p}' \rangle \psi(\mathbf{q}'\mathbf{p}'), \quad (2)$$

where  $\mathbf{q}$  is the relative momentum of two particles,  $\mathbf{p}$  is the momentum of third particle relative to the center of mass of two other particles,  $E$  is the total energy,  $G_0(q, p, E) = m/(mE - q^2 - \frac{3}{4}p^2)$  is the Green's function of noninteracting particles, and  $m$  is the mass of the particle.

By introducing the functions for a given total angular momentum  $\mathcal{Y}_{l\lambda}^{LM}(\hat{q}\hat{p})$

$$\mathcal{Y}_{l\lambda}^{LM}(\hat{q}\hat{p}) = \sum_{m\mu} (l\lambda m\mu | LM) Y_{lm}(\hat{q}) Y_{\lambda\mu}(\hat{p}) \quad (3)$$

(here  $\hat{q}, \hat{p}$  are the unit vectors directed along the vectors  $\mathbf{q}$  and  $\mathbf{p}$ , resp.,  $(l\lambda m\mu | LM)$  is the Clebsch–Gordan coefficient) into (2) we get the equations for partial waves

$$\Psi_{l\lambda}^L(qp) = G_0(qpE) \int q'^2 dq' p'^2 dp' \sum_{l'\lambda'} K_{l\lambda, l'\lambda'}^L(qp, q'p') \Psi_{l'\lambda'}^L(q'p'). \quad (4)$$

The kernel  $K$  has the form

$$K_{l\lambda, l'\lambda'}^L(qp, q'p') \equiv \int \mathcal{Y}_{l\lambda}^{LM*}(\hat{q}\hat{p}) \langle \mathbf{qp} | V | \mathbf{q}'\mathbf{p}' \rangle \mathcal{Y}_{l'\lambda'}^{LM}(\hat{q}'\hat{p}') d\hat{q} d\hat{p} d\hat{q}' d\hat{p}'. \quad (5)$$

As we are solving the bound-state problem of three particles, the solutions of equations (2) and (4) are square-integrable functions. This, in principle, allows us to approximate the square-nonintegrable function  $\langle \mathbf{qp} | V | \mathbf{q}'\mathbf{p}' \rangle$  by a more regular function. Here we use the fact that action of a delta function on a square-integrable function may be replaced by the action of a convergent sequence of regular peaked functions [16]. It is clear that the scattering problem should be handled with more carefully.

Hence, instead of the exact potential  $V = V_{12} + V_{23} + V_{31}$  we introduce the approximate one  $\tilde{V}$

$$\langle \mathbf{qp} | \tilde{V}^N | \mathbf{q}'\mathbf{p}' \rangle = \sum_{i,j=1}^N \langle \mathbf{qp} | V | \chi_i \rangle d_{ij}^{-1} \langle \chi_j | V | \mathbf{q}'\mathbf{p}' \rangle \quad (6)$$

where  $d_{ij} = \langle \chi_i | V | \chi_j \rangle$  and the functions  $\chi_i$  are defined in the six-dimensional space, i.e., expression (6) realizes an approximate separation of the potential. As a result, the problem becomes algebraic. For the application of expansion (6) only the linear independence of functions  $\chi_i$  is required. The functions  $\chi_i$  can, clearly, be introduced in an  $(n < 6)$ -dimensional space. For example, if we introduce them in five-dimensional space, Eq. (4) reduces to a set of one-dimen-

sional integral equations. Let us note that expansion (6) is direct multidimensional analog of the separable expansions of a two-body potential which have been applied extensively to various problems of nuclear physics.

Inserting (6) into (5) we get the approximate kernel

$$\tilde{K}_{l\lambda, l'\lambda'}^L(qp, q'p') = \sum_{i,j} {}^i R_{l\lambda}^L(qp) d_{ij}^{-1} {}^j R_{l'\lambda'}^{L*}(q'p') \quad (7)$$

where

$${}^i R_{l\lambda}^L(qp) = \int \mathcal{W}_{l\lambda}^{LM*}(\hat{q}\hat{p}) \langle \mathbf{qp} | V | \chi_i \rangle d\hat{q} d\hat{p}. \quad (8)$$

Using the approximate kernel instead of the exact one in Eq. (4) we get the expansion in projected matrix elements of the potential for the wavefunction

$$\Psi_{l\lambda}^L(qp) = G_0(qpE) \sum_i {}^i R_{l\lambda}^L(qp) C_i^L(E). \quad (9)$$

The coefficients  $C_i^L(E)$  satisfy the system of algebraic equations

$$C_i^L(E) = \sum_j A_{ij}^L C_j^L(E). \quad (10)$$

Here the matrix  $A_{ij}^L$  is defined as

$$A_{ij}^L = \sum_{k\lambda} d_{ik}^{-1} {}^{kj} I_{l\lambda}^L(E), \quad (11)$$

where

$${}^{ij} I_{l\lambda}^L(E) = \int_0^\infty q^2 dq p^2 dp {}^i R_{l\lambda}^{L*}(qp) G_0(qpE) {}^j R_{l\lambda}^L(qp). \quad (12)$$

The energy of the system is obtained from the condition:

$$\det(1 - A^L(E)) = 0. \quad (13)$$

For the sake of simplicity in the next section we consider states with  $L = 0$ , and besides, restrict ourselves to values  $l = \lambda = 0$ , as we have a short-range potential. In this case expression (8) takes the form

$${}^i R_{00}^0(qp) = \frac{1}{4\pi} \int \langle \mathbf{qp} | V | \chi_i \rangle d\hat{q} d\hat{p}. \quad (14)$$

Introducing the Jacobi coordinates  $\xi, \eta$  for matrix elements  $\langle \mathbf{qp} | V | \chi_i \rangle$  we have

$$\langle \mathbf{qp} | V | \chi_i \rangle = \frac{1}{(2\pi)^3} \int d\xi d\eta e^{i\xi\alpha + i\eta\beta} \langle \xi\eta | \chi_i \rangle [V_{12} + V_{23} + V_{13}]. \quad (15)$$

Here  $\langle \xi \eta | \chi_i \rangle$  are  $\chi_i$  functions in the coordinate representation. Expanding the exponentials in (15) in spherical functions

$$e^{ia\xi} = 4\pi \sum_{lm} i^l j_l(\xi q) Y_{lm}(\hat{q}) Y_{lm}^*(\hat{\xi})$$

and inserting (15) into (14) we get

$$\begin{aligned} {}^i R_{00}^0(qp) &= 4 \int_0^\infty \xi^2 d\xi \int_0^\infty \eta^2 d\eta \int_{-1}^1 dx j_0(\xi q) j_0(\eta p) \langle \xi \eta | \chi_i \rangle \\ &\times [V_{12}(|\xi|) + V_{23}(|\eta + \xi/2|) + V_{13}(|\eta - \xi/2|)] \end{aligned} \quad (16)$$

where  $x = \cos \theta$ ,  $\theta$  is the angle between vectors  $\xi, \eta$ .

### 3. NUMERICAL RESULTS

To see how the separable expansion works in practice, we now specialize our calculation to the two-body potential of the Gaussian type

$$V(\xi) = V_0 e^{-\beta \xi^2}. \quad (17)$$

Parameters  $V_0$  and  $\beta$  are taken from article [6]. For this potential there are known very exact estimates of lower and upper bounds of the binding energy of the three-particle system [6]:

$$-E_T = 9.7813 \pm 0.0024. \quad (18)$$

The functions  $\chi_i$  are chosen also to be of the Gaussian form:

$$\langle \xi \eta | \chi_i \rangle \equiv \chi_i(\xi \eta) = e^{-\alpha_i \xi^2 - \beta_i \eta^2}. \quad (19)$$

Expressions for matrix elements  $d_{ij}, {}^i R_{00}^0(qp)$  and integrals (12) are given in an appendix to this paper. By using the potential (17) and functions (19) integral (16) can be calculated explicitly. Coefficients  ${}^{ij} I_{00}^0(E)$  are expressed as one-dimensional integrals which may be calculated numerically.

The results of binding energy calculations are given in Table I. The choice of the parameters  $a_i$  and  $b_i$  was realized by minimizing energy  $E_i$  with respect to the function  $\chi_i$  without changing the previous  $i - 1$  separable functions. From this table it is clear that even three terms in the expansion of the potential given by expression (6) enable us to find the tritium binding energy with an error not higher than 4%. From the table it is also clear that all the functions  $\chi_i$  become to depend, in practice, only on the six-dimensional radius  $\rho^2 = \xi^2 + \eta^2$ . The wavefunction  $\Psi_{00}^0(qp)$  of the bound state, as seen from equation (A2) of the appendix, depends

TABLE I

$i$	1	2	3	4	5	6	7
$a_i = \alpha_i/\beta$	0,45	0,40	0,35	1,00	0,35	0,45	0,95
$b_i = \beta_i/\beta$	0,55	0,40	0,30	1,00	0,35	0,40	0,95
$-E_i(\text{MeV})$	8,86	8,90	9,46	9,51	9,64	9,68	9,72

on the six-dimensional radius  $R^2 = q^2 + p^2$  only in the limit  $\alpha_i = \beta_i \gg \beta$ , i.e., when the approximate potential  $\tilde{V}$  reproduces well the exact one at distances small compared with the range of the two-body potential.

#### 4. CONCLUSION

Let us briefly discuss the possible range of application of the procedure proposed. By this we mean the rate of convergence of expansion (6) and the practical possibility of calculations of integrals of the type (12) and (16). The use of this method for a sum of two-body potentials with other shapes does not lead to additional complications in the calculation of the ground state of three-body systems. This is because the functions  $\chi_i$  can be chosen such that the integral (16) can be calculated analytically. For instance, for the Yukawa potential the separating functions  $\chi_i$  will be chosen to be of the form  $\chi_i = e^{-\alpha_i \xi - \beta_i \eta}$ .

However, for excited states we may expect a slower convergence of the expansion (6). As the energy increases toward the lowest threshold, the eigenfunction approaches a function of the continuous spectrum, which is non-square-integrable. For the scattering solution the approximation (6) is not justified. In addition, the convergence of expansion (6) can become slower as the number of particles increases, since the dimension of the space in which the separable expansion is carried out becomes larger. This occurs, for example, in the expansion of the bound state wave function in terms of the hyperspherical functions ( $K$ -harmonics) [17].

#### APPENDIX

Here we give the explicit form of matrix elements  $d_{ij}$ , integrals (12) and matrix elements (16), which determine the functional dependence of the bound state wavefunction.

$$d_{ij} \equiv \langle \chi_i | V | \chi_j \rangle = \frac{V_0 \pi^3}{\beta^3} \{ [(a_i + a_j + 1)(b_i + b_j)]^{-3/2} + 16[4(a_i + a_j)(b_i + b_j + 1) + (b_i + b_j)]^{-3/2} \} \quad (\text{A.1})$$

$${}^i R_{00}^0(qp) = \frac{4\pi V_0}{\beta^3} \left[ 2(A_i)^{1/2} e^{-B_i x^2 - C_i y^2} \frac{\text{sh}(A_i x y)}{x y} + D_i e^{-E_i x^2 - F_i y^2} \right]$$

$$x = q/(\beta)^{1/2}, y = p/(\beta)^{1/2}, a_i = \frac{\alpha_i}{\beta}, b_i = \frac{\beta_i}{\beta} \quad (\text{A.2})$$

$$A_i^{-1} = 4a_i(1 + b_i) + b_i \quad D_i^{-1} = 8[(a_i + 1) b_i]^{3/2}$$

$$B_i = (1 + b_i) A_i \quad E_i^{-1} = 4(a_i + 1)$$

$$C_i = \frac{1 + A_i}{4(1 + b_i)} \quad F_i^{-1} = 4b_i$$

$${}^{ij} I_{00}^0(E) = - \frac{16V_0^2 \pi^2 m}{\beta^4} \{ 4{}^{ij} I_{11}(E) + 2[{}^{ij} I_{12}(E) + {}^{ij} I_{12}(E)] + {}^{ij} I_{22}(E) \} \quad (\text{A.3})$$

$${}^{ij} I_{11}(E) = - \frac{(A_i A_j)^{1/2}}{4(3)^{1/2}} \int_{-\pi/2}^{\pi/2} e^{E^{ij} f_1^+(\varphi)} E_i [-E^{ij} f_1^+(\varphi)]$$

$$- e^{E^{ij} f_1^-(\varphi)} E_i [-E^{ij} f_1^-(\varphi)] d\varphi$$

$${}^{ij} f_1^\pm(\varphi) = {}^{ij} B_1 \sin^2 \varphi + \frac{2}{3^{1/2}} {}^{ij} A_1^\pm \sin \varphi \cos \varphi + \frac{4}{3} {}^{ij} C_1 \cos^2 \varphi$$

$${}^{ij} B_1 = B_i + B_j, \quad {}^{ij} A_1^\pm = A_i \pm A_j, \quad {}^{ij} C_1 = C_i + C_j$$

$${}^{ij} I_{12} = \frac{2\pi}{3(3)^{1/2}} (A_j)^{1/2} D_i \left\{ \frac{A_j}{(4/3) A_j^2 + ({}^{ij} B_2 - (4/3) {}^{ij} C_2)^2} \right.$$

$$\times \left( \frac{{}^{ij} B_2 + (4/3) {}^{ij} C_2}{2/3^{1/2} (4 {}^{ij} B_2 {}^{ij} C_2 - A_j^2)^{1/2}} - 1 \right)$$

$$\left. - E \frac{(3)^{1/2}}{2\pi} \int_{-\pi/2}^{\pi/2} e^{E^{ij} f_2(\varphi)} E_i [-E^{ij} f_2(\varphi)] \sin \varphi \cos \varphi d\varphi \right\}$$

$${}^{ij} f_2(\varphi) = {}^{ij} B_2 \sin^2 \varphi + \frac{2}{(3)^{1/2}} A_j \sin \varphi \cos \varphi + \frac{4}{3} {}^{ij} C_2 \cos^2 \varphi$$

$${}^{ij} B_2 = B_i + E_j, \quad {}^{ij} C_2 = C_i + F_j$$

$${}^{ij} I_{22} = \frac{2\pi}{3(3)^{1/2}} \frac{D_i D_j}{(4/3) {}^{ij} C_3 - {}^{ij} B_3} \left\{ \frac{1}{(4/3) {}^{ij} C_3 - {}^{ij} B_3} \left( \frac{(4/3) {}^{ij} C_3 + {}^{ij} B_3}{(4/3) ({}^{ij} B_3 {}^{ij} C_3)^{1/2}} - 1 \right) \right.$$

$$\left. - E \int_0^\infty e^{-(4/3) {}^{ij} C_3 + {}^{ij} B_3} z I_1[(4/3) {}^{ij} C_3 - {}^{ij} B_3] z \frac{dz}{2z + E} \right\}$$

$${}^{ij} B_3 = E_i + E_j, \quad {}^{ij} C_3 = F_i + F_j$$

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